

Locally inequivalent four qubit hypergraph states

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Abstract

Hypergraph states as real equally weighted pure states are important resources for quantum codes of non-local stabilizer. Using local Pauli equivalence and permutational symmetry, we reduce the 32768 four qubit real equally weighted pure states to 28 locally inequivalent hypergraph states and several graph states. The calculation of geometric entanglement supplemented with entanglement entropy confirms that further reduction is impossible for true hypergraph states.

PACS number(s): 03.67.Mn, 03.65.Ud, 03.67.Ac

Keyword(s): hypergraph state; local equivalence; geometric entanglement.

1 Introduction

A real equally weighted pure state (REW) is a superposition of all basis states with real amplitudes and equal probabilities. Recently, it has been systematically proven that REWs have a one-to-one correspondence with the quantum hypergraph states or graph states [1] [2]. Thus a REW can be described by a mathematical hypergraph, namely, a graph where at least one of its edges connecting more than two vertices; or a ‘usual’ graph with two vertex edges. A hypergraph state can be described by a non-local stabilizer, in contrast to a usual graph state which is described by a local stabilizer whose observables are simple tensor products of Pauli matrices. There are a large number of hypergraph states even for a system of a few qubits. Then the local equivalence of hypergraph states is an important problem in applying hypergraph states for quantum information processing. For the local equivalence of arbitrary multipartite pure quantum states, the local polynomial invariants have been introduced [3] and the necessary and sufficient conditions have been proposed [4]. However, a more practical way is to use the entanglement to characterize the local equivalence of states. In this paper, we will study the local equivalence of all the four qubit hypergraph states and classify the states with their geometric measure of entanglement supplemented with bipartite entanglement entropy, both of them are easily calculable.

2 Hypergraph states

A hypergraph $H = (V; E)$ is composed of a set V of n vertices and a set of hyperedges E . The hyperedge set E consists of k -hyperedges (hyperedge connecting k vertices) for $1 \leq k \leq n$. The rank of a hypergraph is the maximum cardinality of its hyperedges. The k -hyperedge neighborhood $N_k(i)$ of the vertex i is defined as $N_k(i) = \{\{i_1, i_2, \dots, i_{k-1}\} | \{i, i_1, i_2, \dots, i_{k-1}\} \in E\}$, where $\{i_1, i_2, \dots, i_{k-1}\}$ is the $k-1$ hyperedge. The neighborhood $N(i)$ of the vertex i is $N(i) = \cup_k N_k(i)$. We also use $N(i)$ to denote the corresponding set of neighbor hyperedges. To associate the hypergraph state with the underlying mathematical hypergraph, we assign each vertex a qubit and initialize each qubit as the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$; each k -hyperedge represents the k -body interaction among the corresponding k qubits. The hypergraph state related to the hypergraph H is defined as

$$|H\rangle = \prod_{k=1}^n \prod_{\{i_1, i_2, \dots, i_k\} \in E} U_{i_1 i_2 \dots i_k} |+\rangle^{\otimes n}, \quad (1)$$

where $U_{i_1 i_2 \dots i_k}$ is the k -qubit control Z gate such that $U_{i_1 i_2 \dots i_k} |11 \dots 1\rangle_{i_1 i_2 \dots i_k} = -|11 \dots 1\rangle_{i_1 i_2 \dots i_k}$, and leaves all the other components of the computational basis unchanged. The rank of a hypergraph state is the rank of its corresponding hypergraph. We will refer rank 3 or higher hypergraph state as true hypergraph state (hereafter a hypergraph state refers to a true hypergraph state). Hypergraph states can also be put into stabilizer formalism [2] as graph states can. The difference is that the stabilizer operators for a graph state are the product of local Pauli operators, while the set of stabilizer operators for a hypergraph state consists nonlocal control phase gate operators. Given a general hypergraph, for any vertex i , the stabilizer operator is defined as

$$K_i = X_i \prod_{k=1}^n \prod_{\{i_1, i_2, \dots, i_{k-1}\} \in N_k(i)} U_{i_1 i_2 \dots i_{k-1}}, \quad (2)$$

where X_i is the Pauli X operator (bit flip) of vertex i . The hypergraph state $|H\rangle$ is stabilized by an Abelian stabilizer group with generator set $\{K_i\}$ [2], namely,

$$K_i |H\rangle = |H\rangle. \quad (3)$$

Considering the local equivalence (denoted as $\stackrel{Local}{=}$) of the hypergraph states, we have $X_i |H\rangle \stackrel{Local}{=} |H\rangle$ since X_i

is a local operator. Using (2) and (3), we have

$$\prod_{k=1}^n \prod_{\{i_1, i_2, \dots, i_{k-1}\} \in N_k(i)} U_{i_1 i_2 \dots i_{k-1}} |H\rangle \stackrel{Local}{=} |H\rangle. \quad (4)$$

Hence applying all the control Z operation containing in the neighborhood of a vertex to a hypergraph state, we obtain a locally equivalent hypergraph state. In the underlying mathematical hypergraph, the transform of hyperedge set produced by X_i is [1]

$$E \rightarrow E' = N(i) \Delta E, \quad (5)$$

where Δ is the symmetric difference, that is, $E \Delta F = E \cup F - E \cap F$. The local equivalence (5) is a very useful tool in classifying the hypergraphs. As an example, suppose there be a four vertex hypergraph H with $E = \{\{1, 2, 3, 4\}, \{1, 2, 3\}\}$, the local equivalent hypergraph H' with $E' = N(4) \Delta E$ can be deduced by applying local operator X_4 of the fourth qubit to the hypergraph state. Then $E' = \{\{1, 2, 3, 4\}\}$. Hypergraph H' is shown as *No.1* in Fig. 1. In fact we can remove all the $n - 1$ hyperedges for an n vertex hypergraph of rank n by applying Pauli X_i operators.

We also have $Z_i |H\rangle \stackrel{Local}{=} |H\rangle$ for Z_i is the Pauli Z operator (phase flip) of vertex i . The transform of hyperedge set produced by Z_i is [1]

$$E \rightarrow E' = \{\{i\}\} \Delta E, \quad (6)$$

that is, a loop is added to (removed from) vertex i when there isn't (is) a loop. The unitary operator corresponds to a loop $\{\{i\}\}$ on vertex i is $U_i = Z_i$ [2]. We can also define hypergraph basis states $|H_C\rangle = Z^C |H\rangle$, where $C = (c_1, \dots, c_n)$ is a bit string with $c_i = 0, 1$, and $Z^C = \otimes_{i=1}^n Z_i^{c_i}$. Then all the local Z equivalent hypergraphs can be written in the form of H_C .

3 Entanglement Measures

The entanglement measures for a multipartite quantum pure states are the Schmidt measure [5], the (logarithmic) geometric measure [6], the relative entropy of entanglement [7], the logarithmic robustness [8] and so on. The later three measures are equal for graph states [9]. Unfortunately, they are not equal for a (true) hypergraph state. Geometric measure is the easiest one to be calculated among these three entanglement measures for the multipartite entanglement. An iterative algorithm was derived for the entanglement of a graph state [10]. The algorithm can also be applied to a hypergraph state after a slight modification. Thus we will use geometric measure to study the entanglement property of four qubit hypergraph states and classify the states by their entanglement values.

For a four qubit hypergraph state, the iterative algorithm of the geometric measure is as follows: let the hypergraph state be $|\psi\rangle$, its closest product state be $|\Phi\rangle =$

$\prod_{i=1}^4 |\phi_i\rangle$, with $|\phi_i\rangle = x_i |0\rangle + y_i |1\rangle$ and $|x_i|^2 + |y_i|^2 = 1$. The overlap amplitude (inner product) of the hypergraph state and its closest product state is $f = \langle \psi | \Phi \rangle$. The Lagrange multiplier method of maximizing $|f|^2$ subject to the conditions $|x_i|^2 + |y_i|^2 = 1$ ($i = 1, \dots, 4$) gives the iterative equations

$$x_i^* = \mathcal{N}_i \frac{\partial f}{\partial x_i}, \quad (7)$$

$$y_i^* = \mathcal{N}_i \frac{\partial f}{\partial y_i}, \quad (8)$$

where \mathcal{N}_i is the normalization. By solving the equations we obtain the closest product state $|\Phi\rangle$. As far as $|\Phi\rangle$ is determined, it follows the overlap amplitude $f = \langle \psi | \Phi \rangle$ of a give hypergraph state $|\psi\rangle$, and the geometric measure of entanglement of $|\psi\rangle$ is

$$E_g = -\log_2 |f|^2. \quad (9)$$

Exact expression of the entanglement values for some of the hypergraph states can also be obtained based on the numeric calculation of the closest product states. As an example, let us consider the *No.12* (in Fig. 2) hypergraph state which is the direct product of $|+\rangle$ with

$$|\psi_3\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle). \quad (10)$$

The later is the hypergraph state of a three qubit hypergraph with a 3-hyperedge and without further two vertex edges. The closest state of $|\psi_3\rangle$ is assumed be $|\phi\rangle^{\otimes 3}$ due to the symmetry of the three qubits, where $|\phi\rangle = x|0\rangle + y|1\rangle$ with normalization $|x|^2 + |y|^2 = 1$. Denote $z = y/x$, then the iterative equation is

$$z^* = (1 + 2z - z^2)/(1 + z)^2. \quad (11)$$

Numeric calculation shows that z eventually converges to a real number in a few steps regardless its randomly chosen initial complex value. Thus we arrive at the algebraic equation $z^3 + 3z^2 - z - 1 = 0$. The solution of which is $z = -1 - \frac{4\sqrt{3}}{3} \cos(\tau + \frac{2\pi}{3}) \approx 0.6751$, where $\tau = \frac{1}{3} \arctan \sqrt{37/27}$. The geometric measure of the state $|\psi_3\rangle$ is

$$E_g = -\log_2 \left| \langle \psi_3 | \phi \rangle^{\otimes 3} \right|^2 \approx 0.5647. \quad (12)$$

The relative entropy of entanglement is $E_r = \min_{\sigma} -\langle \psi_3 | \log_2 \sigma | \psi_3 \rangle$ for a pure state $|\psi_3\rangle$ where σ belongs to the fully separable state set. However, the full separability for a generic three qubit system is unknown. Hence the relative entropy of entanglement is not available. The entanglement is lower bounded by the entanglement of a bipartition of the hypergraph state, this is due to fact that the fully separable state set is the subset of the biseparable state set. The minimization over

a larger set will give a lower value. The bipartite relative entropy of entanglement E_{rbi} is simply the minimal entanglement entropy of all the bipartitions for the pure symmetric state $|\psi_3\rangle$. We have $E_{rbi} = -\text{Tr} \rho \log_2 \rho$ with $\rho = \text{Tr}_{23} |\psi_3\rangle \langle \psi_3| = \frac{3}{4} |+\rangle \langle +| + \frac{1}{4} |-\rangle \langle -|$ being the reduced state by tracing the second and the third qubits, where $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Thus

$$E_r \geq E_{rbi} \approx 0.8113. \quad (13)$$

The relative entropy of entanglement is larger than the geometric measure for the three qubit hypergraph state.

4 Four qubit hypergraph states with a four vertex hyperedge

For the four qubit system, we have REW states

$$|\psi_{REW}\rangle = \frac{1}{4} \sum_{\mu} (-1)^{g(\mu)} |\mu\rangle, \quad (14)$$

where μ is a four bit string and $g(\mu)$ is a Boolean function, i.e. $g: \{0,1\}^{\otimes 4} \rightarrow \{0,1\}$, the coefficient $(-1)^{g(\mu)}$ can be ± 1 for each μ . The state $|\psi_{REW}\rangle$ is uniquely defined by the function g via the signs (either plus or minus) in front of each component μ of the computational basis. It is clear that we have 2^{16} different expressions of the coefficient series. Up to the overall phase, the total number of REW states is 2^{15} . There is a one-to-one correspondence between an REW state and a hypergraph state or graph state. The number of i vertex hyperedges is $C_4^i = \frac{4!}{i!(4-i)!}$. The number of four qubit hypergraph states with a four vertex hyperedge (rank 4) is $N_4 = 2^{14}$, which is a half of the the total number of REW states. Applying the Pauli Z_i operator to a hypergraph state gives rise to a one-vertex hyperedge (namely, the loop) on the i th vertex of the underlying hypergraph. We then remove all the loops of a hypergraph by applying proper Pauli Z operators to obtain the standard hypergraph (hypergraph free of loops). On the other hand, all the three vertex hyperedges within the four vertex hyperedge can be removed by applying the Pauli X_i operators to the hypergraph states. Hence we will consider the standard hypergraphs with two vertex edges and an overall four vertex hyperedge for the four qubit hypergraph states. The number of these hypergraphs is 64 up to local equivalence of the Pauli X_i operators and Z_i operators. Permutational symmetry of the vertices leads to further reduction of 64 hypergraphs to 11 hypergraphs as shown in Fig.1. The geometric measure and bipartite entanglement are listed in Table I.

The exact expression of the entanglement values for some hypergraph states and further properties of the entanglement of hypergraph states with a four qubit edge are shown in Table II.

TABLE I. The geometric measure of entanglement (GE) and bipartite entanglement of four qubit rank 4 hypergraph states. Here BE_2 is the vector of bipartite entanglement of partitions 12|34, 13|24, 14|23. BE_1 is the vector of bipartite entanglement of partitions 1|234, 2|134, 3|124, 4|123. m is the degeneracy with respect to permutational symmetry. The locations of qubits 1, ..., 4 are shown as No.12 in Fig.2. $a = 0.6561$, $b = 1.2624$, $c = 1.6773$, $d = 0.5436$, $e = 0.9544$.

| No. | m | GE | BE_2 | BE_1 |
|-----|-----|--------|--------|---------|
| 1 | 1 | 0.3043 | a,a,a | d,d,d,d |
| 2 | 6 | 0.8157 | a,b,b | e,e,d,d |
| 3 | 3 | 1.4891 | a,c,c | e,e,e,e |
| 4 | 12 | 0.8954 | b,b,b | e,e,d,e |
| 5 | 12 | 1.5261 | b,c,c | e,e,e,e |
| 6 | 4 | 0.8916 | b,b,b | e,e,e,e |
| 7 | 4 | 1.1360 | b,b,b | e,e,d,e |
| 8 | 3 | 1.1732 | c,b,c | e,e,e,e |
| 9 | 12 | 1.4316 | b,c,c | e,e,e,e |
| 10 | 6 | 1.1165 | c,b,c | e,e,e,e |
| 11 | 1 | 1.1726 | b,b,b | e,e,e,e |

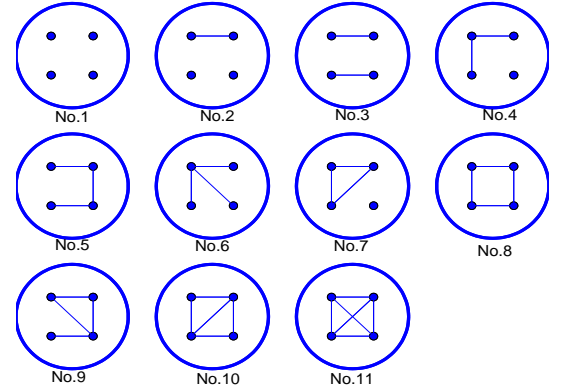


Figure 1: (Color on line) The local inequivalent four vertex hypergraphs with an overall four vertex hyperedge. The four vertex hyperedge is specified by a circle, while the two vertex hyperedges are simply denoted by edges as in [2].

TABLE II. The properties of the four qubit rank 4 hypergraph states. Here ‘PE’ represents for the properties of the geometric entanglement, it can be either exact or numerical. ‘D’ represents for the degeneracies of the closest product states. The closest product state types $|\phi_1\rangle^4$, $|\phi_1\rangle^2|\phi_2\rangle^2$, $|\phi_1\rangle|\phi_2\rangle^2|\phi_3\rangle$, $|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle|\phi_4\rangle$ are denoted with ‘4’, ‘2,2’, ‘1,2,1’, ‘1,1,1,1’, respectively. ‘R/C’ mean the stable values of z_i are real or imaginary.

| No. | PE | D | R/C |
|-----|-----------------------------|---------|-----|
| 1 | Closed form | 4 | R |
| 2 | Numerical | 2,2 | R |
| 3 | Numerical | 2,2 | R |
| 4 | Numerical | 1,2,1 | R |
| 5 | $3 + 2\log_2 \frac{3}{5}$ | 1,1,1,1 | R |
| 6 | Numerical | 2,2 | R |
| 7 | Numerical | 2,2 | C |
| 8 | Closed form | 4 | R |
| 9 | Numerical | 1,2,1 | R |
| 10 | Numerical | 2,2 | R |
| 11 | $5 - \log_2(9 + 3\sqrt{3})$ | 4 | C |

5 Four qubit hypergraph states without a four vertex hyper-edge

Unlike the four qubit hypergraph states with a four vertex hyperedge, the hypergraph states without a four vertex edge have a different locally equivalent classification with respect to Pauli X operators. As can be seen from *No.12* hypergraph shown in Fig. 2, the locally equivalent states are generated by X_1, X_2 and X_3 but not X_4 , where we denote the vertex outside the ellipse as the fourth vertex, while the vertices inside the ellipse are denoted as the first, second and third vertices, respectively. The hypergraph state with underneath hypergraph *No.12* is an eigenstate of X_4 . Thus the number of locally equivalent states produced by Pauli X operators is 8. This is also true for all the other hypergraphs shown in Fig. 2. Together with the Pauli Z local equivalence, we have 128 locally equivalent states for each hypergraph in Fig. 2. The permutational symmetry of vertices gives rise to the degeneracy m in Table III. The total number of hypergraph states with three vertex hyperedges (rank 3) are $N_3 = 128 \times \sum m = 128 \times 120 = 15 \times 2^{10}$. The rank 2 hypergraphs are just graphs. For graph states, the Pauli X operators are equivalent to some other Pauli Z operators and their tensor products when considering local equivalence. Up to local Pauli Z equivalence, the number of the graphs is 64 which is the number $\sum m$ in Table. I. The total number of graphs is $N_2 + N_1 + N_0 = 16 \times 64 = 2^{10}$. Hence the number of hypergraph states of rank 3 together with graph states is $N_3 + N_2 + N_1 + N_0 = 2^{14}$.

TABLE III. The geometric measure of entanglement and bipartite entanglement of four qubit rank 3 hypergraph states. Here BE_2 is vector of the bipartite entanglement of partitions 12|34, 13|24, 14|23. BE_1 is vector of the bipartite entanglement of partitions 1|234, 2|134, 3|124, 4|123. The locations of qubits 1, ..., 4 are shown as No. 12 in Fig.2. $r = 0.8113$, $s = 1.5$, $t = 1.2238$, $u = 1.6009$.

| No. | m | GE | BE_2 | BE_1 |
|-----|-----|--------|--------|---------|
| 12 | 4 | 0.5647 | r,r,r | r,r,r,0 |
| 13 | 12 | 1.5417 | s,s,s | 1,r,1,1 |
| 14 | 12 | 1 | s,s,r | 1,r,1,1 |
| 15 | 4 | 1.5261 | s,s,s | 1,1,1,1 |
| 16 | 6 | 0.6115 | r,t,t | r,r,r,r |
| 17 | 6 | 1.2284 | r,u,u | 1,1,r,r |
| 18 | 12 | 1 | s,t,t | r,r,r,1 |
| 19 | 12 | 1.4150 | s,u,u | 1,1,r,1 |
| 20 | 6 | 1.4569 | s,t,t | r,r,1,1 |
| 21 | 6 | 1.4569 | s,u,u | 1,1,1,1 |
| 22 | 4 | 1 | t,t,t | 1,r,r,r |
| 23 | 12 | 0.6781 | t,t,t | r,r,r,r |
| 24 | 12 | 1.3173 | u,u,t | 1,1,1,r |
| 25 | 4 | 1.4150 | u,u,t | r,1,1,r |
| 26 | 1 | 1.2230 | t,t,t | 1,1,1,1 |
| 27 | 6 | 1.2767 | t,u,u | r,r,1,1 |
| 28 | 1 | 0.8301 | t,t,t | r,r,r,r |

The geometric measure and bipartite entanglement are listed in Table III.

The exact expression of the entanglement values for some hypergraph states and further properties of the entanglement of hypergraph states with three qubit edges are shown in Table IV.

6 Discussion and Conclusion

Each of the rank 4 hypergraphs in section 4 has a characteristic value of geometric entanglement. Hence these 11 hypergraph states are confirmed to be all locally inequivalent due to their different values of geometric entanglement. The entanglement entropy alone is not a good indication for local inequivalent even considering all possible bipartitions. We can see that *No.4* and *No.7* hypergraph states are not discriminated by the series of bipartite entanglement entropy. The spectra of bipartite entanglement entropy for *No.5*, *No.8*, *No.9* and *No.10* hypergraph states are identical or identical under the permutation of qubits. The geometric entanglement of *No.5* hypergraph state coincides with that of *No.15*, however, they are discriminated by their entanglement entropy. The rank 3 hypergraphs in section 5 are all different by their geometric entanglement values except for three cases where further consideration of entangle-

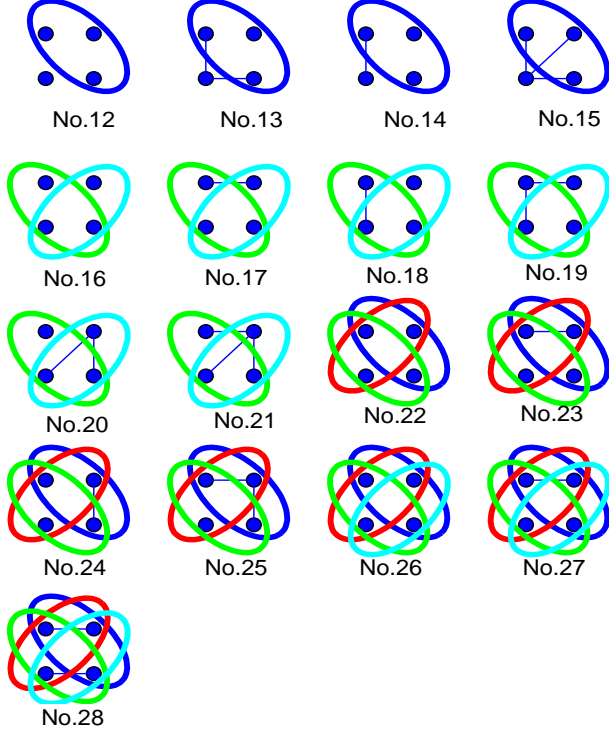


Figure 2: (Color on line) The local inequivalent four vertex hypergraphs without an overall four vertex hyperedge. The three vertex hyperedges are specified by closed curves, while the two vertex hyperedges are simply denoted by edges as in [2].

TABLE IV. The properties of four qubit rank 3 hypergraph states. Here ‘PE’ represents for the properties of the geometric entanglement, it can be either exact or numerical. ‘D’ represents for the degeneracies of the closest product states. The closest product state types $|\phi_1\rangle^4$, $|\phi_1\rangle|\phi_2\rangle^3$, $|\phi_1\rangle^2|\phi_2\rangle^2$, $|\phi_1\rangle|\phi_2\rangle^2|\phi_3\rangle$, $|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle|\phi_4\rangle$ are denoted with ‘4’, ‘1,3’, ‘2,2’, ‘1,2,1’, ‘1,1,1,1’, respectively. ‘R/C’ mean the stable values of z_i are real or imaginary.

| No. | PE | D | R/C |
|-----|------------------------------|-------|-----|
| 12 | Closed form | 1,3 | R |
| 13 | Numerical | 1,2,1 | R |
| 14 | 1 | 1,3 | R |
| 15 | $3 + 2 \log_2 \frac{3}{5}$ | 1,3 | R |
| 16 | $4 - 2 \log_2(1 + \sqrt{5})$ | 2,2 | R |
| 17 | $2.5 - \log_2(1 + \sqrt{2})$ | 2,2 | C |
| 18 | 1 | 1,3 | R |
| 19 | $3 - \log_2 3$ | 1,2,1 | R |
| 20 | $4 - 2 \log_2(1 + \sqrt{2})$ | 1,2,1 | R |
| 21 | $4 - 2 \log_2(1 + \sqrt{2})$ | 2,2 | R |
| 22 | 1 | 1,3 | R |
| 23 | $3 - \log_2 5$ | 1,3 | R |
| 24 | Numerical | 1,2,1 | R |
| 25 | $3 - \log_2 3$ | 1,2,1 | R |
| 26 | $6 - 2 \log_2(3 + \sqrt{5})$ | 4 | R |
| 27 | Numerical | 2,2 | R |
| 28 | $4 - 2 \log_2 3$ | 4 | R |

ment entropy is necessary. The three cases are that the values of geometric entanglement of *No.19* and *No.25* hypergraph states are equal; the values of geometric entanglement of *No.20* and *No.21* hypergraph states are equal; the values of geometric entanglement of *No.14*, *No.18* and *No.22* hypergraph states are also equal. Further discriminations are carried out by their bipartition entanglement entropy.

We conclude that there are 28 locally inequivalent four qubit hypergraph states, 11 of them are rank 4, and 17 of the them are rank 3. They can be discriminated by geometric entanglement supplemented with bipartition entanglement entropy. Local Pauli equivalence together with permutation symmetry are enough in discriminating inequivalent four qubit hypergraph states (it may not be true for hypergraph states with more qubits). Further local Clifford equivalence such as local complementary transform is not necessary (it is also limited to four qubit hypergraph states), although it is a powerful tool in discriminating local inequivalent graph states. We have given a complete classification of the real equally weighted four qubit pure states.

Note added: After submission of this work, we became aware of a recent preprint by O. Gühne *et al* [11], which shows a similar result.

Acknowledgement

We thank the National Natural Science Foundation of China (Grant Nos. 11375152, 60972071) for support.

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